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# Structured Pseudospectra in Structural Engineering

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## Abstract

This paper presents a new method for computing the pseudospectra of a matrix that respects a prescribed sparsity structure. The pseudospectrum is defined as the set of points in the complex plane to which an eigenvalue of the matrix can be shifted by a perturbation of a certain size. A canonical form for sparsity preserving perturbations is given and a computable formula for the corresponding structured pseudospectra is derived. This formula relates the computation of structured pseudospectra to the computation of the structured singular value (ssv) of an associated matrix. Although the computation of the ssv in general is an NP-hard problem, algorithms for its approximation are available and demonstrate good performance when applied to the computation of structured pseudospectra of medium-sized or highly sparse matrices.

The method is applied to a wing vibration problem, where it is compared with the matrix polynomial approach, and to the stability analysis of truss structures. New measures for the vulnerability of a truss structure are proposed, which are related to the ‘distance to singularity’ of the associated stiffness matrix.

*Key words:* matrices, pseudospectra, structural engineering, vulnerability, sensitivity analysis

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## 1 Introduction

Spectral analysis plays an important role in many areas of applied mathematics and engineering. For example, the stability of a system can often be determined by considering the spectrum of an associated matrix. However, if the matrix is non-normal, that is, if it has non-orthogonal eigenvectors, then its spectrum can be highly sensitive to perturbations. Therefore, to investigate the robust stability of a system one should take into account perturbations of the matrix.

In many cases it is not meaningful to investigate sensitivity with respect to all possible perturbations. Indeed, many problems possess an inherent structure that should be respected in their analysis. As a specific class of examples consider sparse matrices, which appear in a variety of problems in structural analysis, digital circuit simulation, finite element analysis or in the finite difference discretisation of almost any system of partial differential equations. Sparsity is often defined by the structure of the underlying problem and therefore only perturbations with the same sparsity structure are of real interest in a sensitivity or robustness analysis.

In this paper we will present a method for analysing the sensitivity of eigenvalues of a matrix with respect to a set of *admissible perturbations*. Our notion of admissibility is designed to cover perturbations that preserve a prescribed sparsity structure. In contrast to the well-known results for arbitrary perturbations (see below) our refined approach requires rather elaborate mathematical derivations.

### 1.1 Pseudospectra of matrices

The *pseudospectra* of a matrix as popularised by Trefethen [1,2], describe the behaviour of its eigenvalues under perturbations. For each  $\varepsilon \geq 0$ , the  $\varepsilon$ -pseudospectrum  $\Lambda_\varepsilon(A)$  of a matrix  $A \in \mathbb{C}^{n \times n}$  is the set of all numbers  $z \in \mathbb{C}$  that are eigenvalues of  $A + P$  for some perturbation  $P \in \mathbb{C}^{n \times n}$  of size  $\varepsilon$ . The size of the matrix  $P$  is measured in its 2-norm  $\|P\|$ , which can be defined as the largest singular value of  $P$ .

**Definition 1.1** For a matrix  $A \in \mathbb{C}^{n \times n}$  let  $\Lambda(A)$  denote its spectrum and let  $\|A\|$  denote its 2-norm. Then

$$\Lambda_\varepsilon(A) := \bigcup_{\|P\| \leq \varepsilon} \Lambda(A + P).$$

Numbers in  $\Lambda_\varepsilon(A)$  are called *pseudo-eigenvalues* of  $A$ .

Pseudospectra have been used to measure robustness and to predict transient instability in a variety of areas, such as fluid dynamics and control theory, see for instance [2,3] and references therein.

An  $\varepsilon$ -pseudoeigenvalue  $z$  of  $A$  is characterised by the fact that the minimal singular value  $\sigma_{\min}$  of  $zI - A$  is not bigger than  $\varepsilon$ , [3]

$$\Lambda_\varepsilon(A) := \{z \in \mathbb{C} : \sigma_{\min}(zI - A) \leq \varepsilon\}.$$

(Here  $I$  denotes the  $n \times n$  identity matrix.) This characterisation allows the computation of  $\Lambda_\varepsilon(A)$  as level sets of  $f(z) := \sigma_{\min}(zI - A)$  on a grid in the com-

plex plane or the computation of the boundaries of  $\Lambda_\varepsilon(A)$  using continuation methods, see [3].

Many problems possess a certain structure, which should be taken into account in stability and robustness considerations. Classic examples are linear second order systems of ordinary differential equations (ODEs)  $M\ddot{q} + C\dot{q} + Kq = 0$ , where  $M$ ,  $C$ , and  $K$  are  $m \times m$ -matrices and  $q \in \mathbb{R}^m$ . In vibration theory  $M$  is a mass matrix,  $C$  a damping matrix, and  $K$  is a stiffness matrix. Rewritten as a first order system, this second order equation becomes  $\dot{x} = Ax$  with  $x = (q, \dot{q}) \in \mathbb{R}^{2m}$  and

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix}.$$

Usually, one is interested in the sensitivity of the eigenvalues of  $A$  under perturbation of  $M$ ,  $C$  and  $K$ . Instead of considering arbitrary perturbations one should therefore only choose perturbations that preserve the block structure of  $A$  (and in particular the structure of its first  $m$  rows).

One way to achieve this is by considering the associated *matrix polynomial* or  *$\lambda$ -matrix*, [4]

$$Q(\lambda) = M\lambda^2 + C\lambda + K. \quad (1)$$

An *eigenvalue* of (1) is a number  $\lambda \in \mathbb{C}$  for which it becomes singular, that is, for which there exists a vector  $q \neq 0$  such that  $(M\lambda^2 + C\lambda + K)q = 0$ . If  $M$  is non-singular, the eigenvalues of  $Q$  coincide with those of  $A$ . To study the sensitivity of eigenvalues of such systems, the notion of pseudospectra of matrix polynomials have been introduced by Tisseur and Higham [5].

Motivated by problems in control theory another approach for the treatment of structured perturbations has been proposed by Hinrichsen and Pritchard [6], where for some matrix  $A$  perturbations of the form  $A + DPE$  are considered. Here, the matrices  $D \in \mathbb{C}^{n \times k}$  and  $E \in \mathbb{C}^{l \times n}$  are fixed, and  $P \in \mathbb{C}^{k \times l}$  is the perturbation matrix. By choosing suitable matrices  $D$  and  $E$  it is possible to perturb only single entries or single blocks in  $A$ . Pseudospectra with respect to this class of perturbations have been introduced by Hinrichsen and Kelb in [7], where the notion of *spectral value sets* is used. Their main result is of interest to us, since it gives a computable formula for the spectral value sets.

**Lemma 1.1** ([7]) *Let  $G(z) := E(zI - A)^{-1}D$ , and choose  $z \in \mathbb{C} \setminus \Lambda(A)$ . Let  $\|\cdot\|$  denote the 2-norm in  $\mathbb{C}^{k \times l}$ . Then*

$$\|G(z)\|^{-1} = \min\{\|P\| : P \in \mathbb{C}^{k \times l}, z \in \Lambda(A + DPE)\},$$

where  $0^{-1} = \infty$  and  $\min \emptyset := \infty$ .

In this article we will deal with the computation of pseudospectra with respect to perturbations of certain entries of a matrix  $A$ . This leads to a generalisation of the results in [6,7]. Our particular interest lies in the case where  $A$  is sparse and the sparsity structure is to be preserved under perturbations. Roughly speaking this means that only non-zero elements of  $A$  should be perturbed. As indicated above this class of problems is of interest in a number of applications. As a particular example we will deal with matrix methods in the linear analysis of truss structures.

### 1.2 *Sparse matrices in structural engineering*

Let us consider two-dimensional truss structures, consisting of nodes that are connected by bars. With each node there are two associated degrees of freedom, unless there are constraints. Let  $n$  equal the total number of degree of freedoms of the structure, that is, twice the number of nodes minus the number of imposed constraints. Let  $f$  be a vector of external forces applied to the nodes, and let  $\delta$  be the vector of displacements of the degrees of freedom. For small deformations the relation between forces and displacements can be linearly approximated as

$$K\delta = f, \quad (2)$$

where  $K \in \mathbb{R}^{n \times n}$  is the *stiffness matrix* of the structure, see [8,9]. It is important that  $K$  can be decomposed as  $K = V^T M V$ , where the matrix  $M \in \mathbb{R}^{l \times l}$  is a diagonal matrix with the (positive) elastic constants of the  $l$  bars of the structure on the diagonal. The matrix  $V$  can be viewed as an incidence matrix, whose non-zero entries also describe the two-dimensional geometry. We find that the connectivity of the structure translates to a sparsity structure of the stiffness matrix  $K$ . More precisely, in  $K$  each element (bar) is typically represented by four blocks of non-zero elements, see Figure 1. The dimension of the single blocks will depend on the number of constraints at the corresponding nodes. If there are no constraints, then the situation is as depicted in Figure 1.

### 1.3 *Robustness of truss structures*

We are interested in describing the robustness or *vulnerability* [10] of a truss structure. One way of treating this problem is to investigate how much a structure has to be damaged before it collapses. Such investigations of structural vulnerability have been undertaken in [10,11]. There, based on a theory of form and connectivity, an algorithmic approach has been developed to identify weak links in a structure.

Our goal is to extract similar information directly from the stiffness matrix  $K$ .

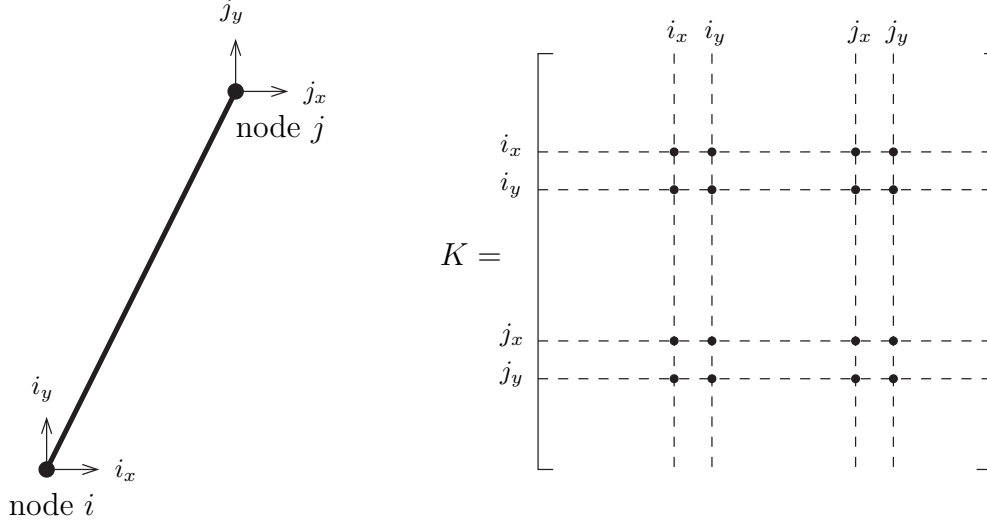


Fig. 1. Connections between nodes in a truss structure give rise to four  $2 \times 2$  blocks in the associated stiffness matrix  $K$

It is well-known that a structure is unstable if its stiffness matrix is singular, [8]: If in (2) the matrix  $K$  is singular, then there will exist a nontrivial solution to  $K\delta = 0$ , such that a zero vector of external forces will lead to a non-zero vector of displacements. That is, (parts of) the structure will collapse or *become a mechanism*. Moreover, applying damage to a structure results in a new stiffness matrix, and therefore this damage can be viewed as a perturbation to  $K$ . We therefore propose to use the ‘distance to singularity’ of  $K$ , that is, the size of the smallest perturbation that leads to  $K$  becoming singular, as a measure of the robustness of a structure.

If one considers general perturbations of  $K$ , then a suitable distance is given in terms of matrix norms as in Definition 1.1. Since  $K = V^T M V$  is positive semi-definite, this distance to singularity is equal to the smallest eigenvalue of  $K$ , see for instance [12,2].

It is however reasonable to assume that damage to a truss structure does not introduce new elements between nodes. Therefore, instead of looking at the distance to singularity in the space  $\mathbb{C}^{n \times n}$  one might also restrict to the space of matrices with the same sparsity structure as  $K$ . Similarly, if one is interested in the sensitivity under variation of material properties of the structure (for instance reduced cross sectional areas of the members, caused by corrosion), then one should study perturbations of the material matrix  $M$  only. We will see that these two problems can be treated along the same lines and moreover, that both the general and the ‘structured’ analysis can give valuable information about the vulnerability of a structure.

Finally, one has to introduce an appropriate notion of distance in the space of sparsity preserving perturbations. Since we are interested in component-

wise modifications of  $K$  we suggest that distances should also be measured componentwise. In fact, it would be inappropriate to measure, for example, corrosion in terms of the 2-norm of a matrix, since the bars in the structure - exposed to the same weather conditions - would corrode according to the same probability distribution.

These (rather imprecise) ideas lead to the notion of *structured pseudospectra*, introduced in Section 2. First, a suitable description of sparsity preserving perturbations is derived, and it is shown how pseudospectra with respect to such perturbations can be computed. This amounts to computing the so-called *structured singular value* of a matrix. We will give a short introduction to this. In Section 3 the theory is applied to examples. We first consider a well-studied model from linear vibration theory. Our main interest here lies in comparing our approach of a ‘structure preserving’ analysis to the classic way of using matrix polynomials. Afterwards we return to the analysis of two-dimensional truss structures and discuss the before mentioned measures of vulnerability for several examples. Finally, Section 4 draws conclusions.

## 2 Admissible perturbations and structured pseudospectra

Let  $A = (A_{ij})$  be an  $n \times n$ -matrix. We are interested in perturbations of  $A$  that leave certain of its entries fixed. For this we choose a non-negative matrix  $W = (W_{ij}) \in \mathbb{C}^{n \times n}$  and consider the class of admissible perturbations

$$\mathbf{P} := \left\{ P = (P_{ij}) \in \mathbb{C}^{n \times n} : P_{ij} = 0, \text{ if } W_{ij} = 0 \right\}.$$

This description of admissible perturbations is very general. For example, by choosing  $W = |A|$ , that is  $W_{ij} = |A_{ij}|$  for all  $i, j$ , we find that  $W_{ij} = 0$  if and only if  $A_{ij} = 0$ , and thus matrices in  $\mathbf{P}$  yield *componentwise perturbations*, that is, only non-zero entries of  $A$  are perturbed. In addition, we will use the non-zero entries of  $W$  to associate weights to perturbations of single elements of  $A$ .

We first derive a canonical form for perturbations in  $\mathbf{P}$ . Let

$$e_i = (0, \dots, 0, 1, 0, \dots, 0)^T$$

denote the  $i$ -th unit vector in  $\mathbb{R}^n$  and observe that the product  $e_i e_j^T$  yields a matrix  $E = (E_{kl})$  whose only non-zero entry is  $E_{ij} = 1$ . A perturbation of the entry  $A_{ij}$  in  $A$  by some  $P_{ij}$  can therefore be written as  $A + P_{ij} e_i e_j^T$ . We will use the  $W_{ij}$  to associate weights to the perturbation of the entries  $A_{ij}$  of  $A$

and consequently consider weighted admissible perturbations of  $A$  of the form

$$A + \sum_{i=1}^n \sum_{j=1}^n W_{ij} \cdot P_{ij} \cdot e_i e_j^T. \quad (3)$$

In the following we only consider those pairs of  $i, j$  with  $W_{ij} \neq 0$ . We now want to rewrite the above sum in a form similar to the one studied in [6,7]. First, let  $p \in \mathbb{C}^n$  be the vector of the  $P_{ij}$  (stacked row-wise) and set  $\Delta = \text{diag}(p)$ . The sum in Equation (3) can then be written as a product of matrices by introducing suitable matrices

$$\begin{aligned} D &= [W_{11} \cdot e_1, W_{12} \cdot e_1, \dots, W_{1n} \cdot e_1, W_{21} \cdot e_2, \dots, W_{nn} \cdot e_n], \\ E &= [e_1^T, e_2^T, \dots, e_n^T, e_1^T, \dots, e_n^T]^T. \end{aligned}$$

If the  $i$ -th row in  $W$  has  $l$  non-zero entries then the vector  $e_i$  appears  $l$  times as a column in  $D$ . The corresponding row-vectors in  $E$  are chosen according to the position of the non-zero entries in this row of  $W$ . Using  $D$  and  $E$  we can write perturbations of the form (3) as

$$A + D\Delta E. \quad (4)$$

Assuming that  $W$  has  $m$  non-zero entries, we find  $\Delta \in \mathbb{C}^{m \times m}$ ,  $D \in \mathbb{C}^{n \times m}$ ,  $E \in \mathbb{C}^{m \times n}$ . Note that the perturbation matrix  $\Delta \in \mathbb{C}^{m \times m}$  has to be a diagonal matrix. Before we introduce structured pseudospectra we want to illustrate the foregoing with a simple example.

**Example:** Let  $A$  be some  $3 \times 3$  matrix and consider the weight matrix

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Weighted admissible perturbations with respect to  $W$  only perturb the entries  $A_{11}, A_{21}, A_{32}$  and  $A_{33}$  of  $A$ , and, moreover, the perturbation of  $A_{33}$  is given a double weight compared to the others. (This could be motivated by parameters of the system that appear in these positions.) Admissible perturbations of  $A$  are given by

$$A + P_{11} \cdot e_1 e_1^T + P_{21} \cdot e_2 e_1^T + P_{32} \cdot e_3 e_2^T + 2P_{33} \cdot e_3 e_3^T,$$



or

$$A + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} P_{11} & 0 & 0 & 0 \\ 0 & P_{21} & 0 & 0 \\ 0 & 0 & P_{32} & 0 \\ 0 & 0 & 0 & P_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

from which the matrices  $D$  and  $E$  can be seen.

### 2.1 The structured pseudospectrum

For the definition of the  $\varepsilon$ -pseudospectrum with respect to a weight matrix  $W$  it is useful to observe that we are interested in the sensitivity of the eigenvalues of  $A$  when certain elements of the matrix are modified. Instead of measuring the size of such perturbations in terms of matrix norms as in Definition 1.1 above it is therefore more appropriate to use the actual size of the modifications of elements of  $A$  as a measure for the size of the perturbation. Using the canonical form (4) of perturbations with  $\Delta = \text{diag}(p)$  we have  $\|\Delta\| = \max_{i,j} |P_{ij}|$ . Consequently, denoting the linear space of (complex)  $m \times m$ -diagonal matrices by  $\mathbf{\Delta}$  we define the structured  $\varepsilon$ -pseudospectrum  $\Lambda_\varepsilon^W$  of  $A$  with respect to  $W$  as follows.

#### Definition 2.1

$$\Lambda_\varepsilon^W(A) = \bigcup_{\substack{\Delta \in \mathbf{\Delta} \\ \|\Delta\| \leq \varepsilon}} \Lambda(A + D\Delta E).$$

We note that the  $\varepsilon$ -pseudospectrum depends on the chosen weights  $W_{ij}$ , which according to the above construction enter the definition through the matrix  $D$ .

We next derive a formula for the computation of  $\Lambda_\varepsilon^W(A)$ , which uses the concept of the so-called structured singular value (ssv) or  $\mu$ -function, [13]. For a matrix  $B \in \mathbb{C}^{m \times m}$  we define its ssv  $\mu_{\mathbf{\Delta}}(B)$  by

$$\mu_{\mathbf{\Delta}}(B) := (\min\{\|\Delta\| : \Delta \in \mathbf{\Delta} \text{ and } \det(I - \Delta B) = 0\})^{-1}.$$

Note that this definition of the ssv can be made for general closed subsets  $\widetilde{\mathbf{\Delta}}$  of  $\mathbb{C}^{m \times m}$ . (If no  $\Delta \in \widetilde{\mathbf{\Delta}}$  exists that makes  $I - \Delta B$  singular, then one has to set  $\mu_{\widetilde{\mathbf{\Delta}}}(B) = 0$ .) In particular, if  $\widetilde{\mathbf{\Delta}} = \mathbb{C}^{m \times m}$  we have  $\mu_{\widetilde{\mathbf{\Delta}}}(B) = \sigma_{\max}(B)$ , that is, the function yields the largest singular value of  $B$ . For this reason  $\mu_{\mathbf{\Delta}}(B)$  is called the *structured singular value* of  $B$ , see [13].

Our main result shows how the pseudospectra of  $A$  with respect to the class of admissible perturbations  $\mathbf{P}$  can be computed.

**Lemma 2.1** *Assume the matrices  $D$  and  $E$  are formed as described above and let  $\Delta \subset \mathbb{C}^{m \times m}$  denote the space of diagonal matrices. For  $z \in \mathbb{C} \setminus \Lambda(A)$  let  $G(z) = E(zI - A)^{-1}D$  and set  $F(z) := \mu_{\Delta}(G(z))$ . Then*

$$F(z)^{-1} = \min\{\|\Delta\| : \Delta \in \Delta : z \in \Lambda(A + D\Delta E)\}.$$

**Proof.** Note first that  $\mu_{\Delta}(B) = (\min\{\|\Delta\| : \Delta \in \Delta \text{ and } 1 \in \Lambda(\Delta B)\})^{-1}$ . Therefore the assertion follows from the next equivalence

$$z \in \Lambda(A + D\Delta E) \Leftrightarrow 1 \in \Lambda(\Delta G(z)), \quad z \in \mathbb{C} \setminus \Lambda(A). \quad (5)$$

To prove (5) we first choose  $z \in \Lambda(A + D\Delta E)$ ,  $z \notin \Lambda(A)$  with eigenvector  $x$ . Then  $0 = (zI - A)(I - (zI - A)^{-1}D\Delta E)x$  and thus  $x = (zI - A)^{-1}D\Delta E x$ . Multiplying this equation by  $\Delta E$  shows  $\Delta E x = \Delta G(z)\Delta E x$ , and hence,  $1 \in \Lambda(\Delta G(z))$ . For the proof of the converse one simply has to follow the arguments backwards.  $\square$

**Corollary 2.2**  $\Lambda_{\varepsilon}^W(A) = \{z \in \mathbb{C} : F(z) \geq \varepsilon^{-1}\}$ .

A similar result can be derived if only *real* structured perturbations of  $A$ , that is  $A + D\Delta E$  with  $\Delta \in \mathbb{R}^{m \times m}$ , are allowed. Indeed, Lemma 2.1 is also valid in this case, provided that  $F_{\mathbb{R}}(z) := \mu_{\Delta_{\mathbb{R}}}(G(z))$  is used instead of  $\mu_{\Delta}(G(z))$ , where  $\Delta_{\mathbb{R}} \subset \mathbb{R}^{m \times m}$  denotes the space of real diagonal matrices. This is of interest, since most matrices in structural dynamics and, in particular, the matrices in the examples studied in Section 3 are real and are subject to real perturbations only. Unfortunately, the computation (or approximation) of  $F_{\mathbb{R}}(z)$  is much more complicated than that of  $\mu_{\Delta}(G(z))$ , see also Section 2.2 below. This means that the result about structured pseudospectra with respect to real perturbations can hardly be used in applications. We therefore concentrate on pseudospectra under complex admissible perturbations of the matrix  $A$ .

## 2.2 Computational aspects

Lemma 2.1 relates the computation of structured pseudospectra to the computation of the ssv of  $E(zI - A)^{-1}D$ . By determining this value for points  $z$  on a grid in the complex plane it is straightforward to derive approximations of the structured pseudospectra of the matrix  $A$ . Unfortunately, there is no general method for computing the ssv of a matrix. Even worse so, the computation of the ssv is known to be an NP-hard problem, which makes it computationally intractable for matrices of large dimension, see [14] and references therein.

Instead of computing the ssv directly research interests have therefore shifted to finding approximations of this value. It is important to note that most methods yield approximations for the complex ssv, and that the methods for approximating the real ssv, needed for computing  $F_{\mathbb{R}}(z)$ , show a significantly poorer performance.

For the computation of structured pseudospectra we note first that it is straightforward to compute subsets of the actual pseudospectra, using random perturbation matrices  $\Delta$  with prescribed norm. On the other hand, several methods for approximating the (complex) ssv of a given matrix are included in the Robust Control Toolbox in MATLAB [15]. These methods compute upper bounds for the ssv.

Lemma 2.1 thus shows that, applied to pseudospectra computations, the use of these methods gives supersets of the exact pseudospectra. Therefore the combination of both approaches yields bounds for structured pseudospectra of a matrix. The quality of these bounds decreases with the size of the involved matrices. Since in our approach this size mainly depends on the number of elements in  $A$  that should be perturbed our method works well for medium-sized or highly-sparse matrices.

### 3 Numerical examples

In this section we apply the foregoing theory to two different classes of examples. We first consider a second order system of ODEs arising from the analysis of oscillations of a wing in an airstream. Finally, we return to structural engineering problems and investigate the vulnerability of two-dimensional truss structures. We apply structured pseudospectra to introduce a measure for vulnerability that only uses information from the associated stiffness matrices.

#### 3.1 Vibrations of an aircraft wing

The system of second order ODEs

$$M\ddot{q} + C\dot{q} + Kq = 0,$$

with

$$M = \begin{pmatrix} 17.6 & 1.28 & 2.89 \\ 1.28 & 0.824 & 0.413 \\ 2.89 & 0.413 & 0.725 \end{pmatrix}, \quad C = \begin{pmatrix} 7.66 & 2.45 & 2.1 \\ 0.23 & 1.04 & 0.223 \\ 0.6 & 0.756 & 0.658 \end{pmatrix},$$

$$\text{and } K = \begin{pmatrix} 121 & 18.9 & 15.9 \\ 0 & 2.7 & 0.145 \\ 11.9 & 3.64 & 15.5 \end{pmatrix}$$

serves as a linear approximation for oscillations of a wing in an airstream. The sensitivity of its eigenvalues has been studied in [5,16]. There, in order to respect the second-order structure of the problem pseudospectra of the associated matrix polynomial  $Q(\lambda) = M\lambda^2 + C\lambda + K$  were considered. We will use the example to compare this classic approach with our notion of structured pseudospectra.

Let us first study general, ‘unstructured’ pseudospectra for this problem. For this we rewrite the equation as a first order system  $\dot{x} = Ax$  with

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -11.85 & -0.82 & 8.76 & -0.89 & 0.03 & 0.11 \\ 4.14 & -1.57 & 20.20 & -0.36 & -1.01 & 0.32 \\ 28.46 & -0.87 & -67.83 & 2.92 & -0.59 & -1.51 \end{pmatrix}.$$

Note that  $\Lambda(A) = \{-0.88 \pm i8.44, 0.09 \pm i2.52, -0.92 \pm i1.76\}$ . Pseudospectra of  $A$  can be computed as level sets of the function  $f(z) = \sigma_{\min}(zI - A)$ , [3]. In panel (a) of Figure 2 we show boundaries of  $\varepsilon$ -pseudospectra for certain values of  $\varepsilon$ . The sensitivity of each of the eigenvalues to perturbations can be seen to be approximately the same.

In order to analyse how the sensitivity is affected if we preserve the second order structure we first consider the matrix polynomial  $Q$  and determine its pseudospectra. Let us explain how this can be done, following [5]. Consider perturbations of  $Q$  of the form

$$Q_P(\lambda) = (M + P_2)\lambda^2 + (C + P_1)\lambda + (K + P_0),$$

with the  $P_i$  being perturbation matrices. Moreover, we introduce a vector of nonnegative *weights*  $w = (w_0, w_1, w_2)$ . Then the  $\varepsilon$ -pseudospectrum of  $Q(\lambda)$  is defined to be

$$\Lambda_{\varepsilon,w}(Q) = \{\lambda \in \mathbb{C} : \det Q_P(\lambda) = 0, \|P_i\| \leq \varepsilon w_i, i \in \{0, 1, 2\}\}, \quad (6)$$

where  $\|\cdot\|$  denotes again the 2-norm. It is shown in [5] that similar to the case of a single matrix pseudospectra of a matrix polynomial can be computed in a convenient way.

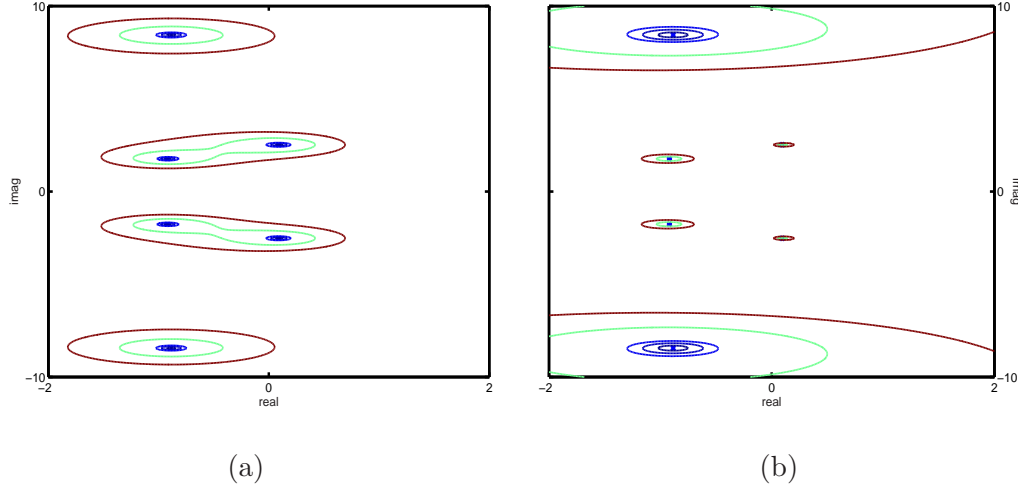


Fig. 2. Pseudospectra of matrix  $A$ , part (a), and of the associated matrix polynomial  $Q$ , part (b). The figures contain boundaries of the  $\varepsilon$ -pseudospectra in the complex plane for  $\varepsilon \in \{0.01, 0.02, 0.03, 0.1, 0.2\}$ . For the computation of the pseudospectra of  $Q$  we have set all weights  $w_i = 1$ ; see text for details.

**Lemma 3.1** ([5], Lemma 2.1) *Let  $q_w(\lambda) = w_2\lambda^2 + w_1\lambda + w_0$ . Then*

$$\Lambda_{\varepsilon,w}(Q) = \{\lambda \in \mathbb{C} : \sigma_{\min}(Q(\lambda)) \leq \varepsilon q_w(|\lambda|)\}.$$

This result allows us to compute pseudospectra of the above polynomial over a grid in the complex plane. In part (b) of Figure 2 contours of pseudospectra are shown. Here the size of the perturbations  $P_i$  is measured in an absolute sense by setting  $w_2 = w_1 = w_0 = 1$ . It can be observed that the eigenvalues with the largest and smallest imaginary part are most sensitive under perturbations of  $M$ ,  $C$  and  $K$ . (Similar results have been obtained for other choices of the weights  $w_i$ .)

Let us compare these results with the computation of structured pseudospectra of  $A$  using the methods developed in Section 2 above. In order to respect the second-order structure of the system only the three bottom rows of  $A$  should be perturbed. Choosing equal weights for all entries of the bottom rows the

weight matrix  $W$  consequently reads

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \quad (7)$$

Structured pseudospectra of  $A$  can be computed as level sets of the function  $F$ , introduced in Lemma 2.1, evaluated on a grid in the complex plane. For the computation the MATLAB function `psv` has been used, which computes an upper bound for the ssv using the Perron optimal diagonal scaling [17,15]. (Further computations have shown the results to be essentially independent of the employed method for computing the structured singular values.) For certain values of  $\varepsilon$  the boundaries of  $\varepsilon$ -pseudospectra are shown in part (a) of Figure 3.

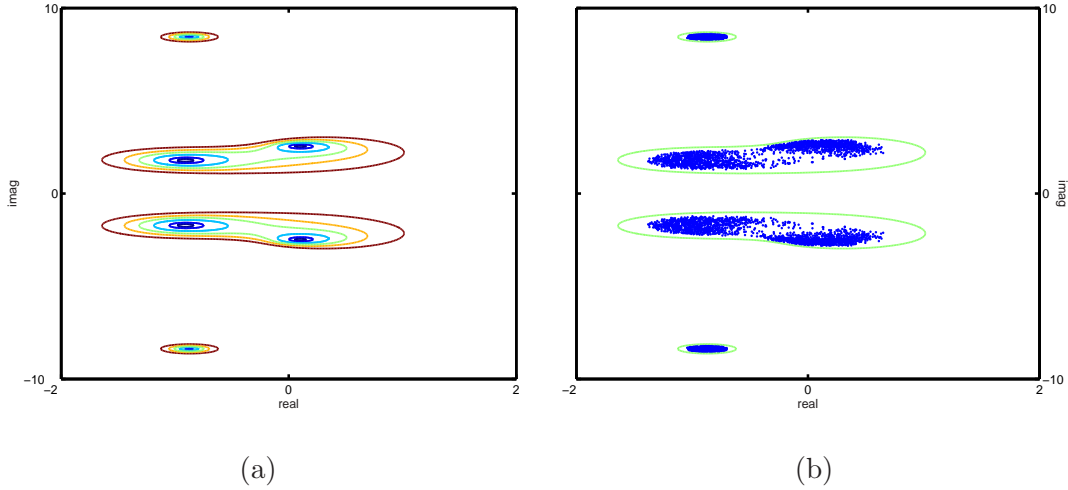


Fig. 3. Structured pseudospectra of  $A$  with respect to the matrix  $W$  in (7). Panel a) contains approximations to the boundaries of  $\varepsilon$ -pseudospectra for  $\varepsilon \in \{0.05, 0.1, 0.2, 0.4, 0.6\}$ . The computation is based on Lemma 2.1. Panel b) shows eigenvalues of random structured perturbations of size  $\varepsilon = 0.6$  together with boundary of the  $\varepsilon = 0.6$ -pseudospectrum.

One should not expect that the plots agree exactly with Figure 2(a), since the size of the perturbations is measured differently in the employed methods. However, it is surprising that there are also qualitative differences. Indeed, inspection of panel (a) of Figure 3 shows that now the eigenvalues closest to the real axis are more sensitive to perturbations. Since we only compute an

upper bound to the ssv (and therefore supersets of the actual pseudospectra) one could argue that this remarkable effect is caused by a bad approximation. But Figure 3(b) shows that this is not the case. In this plot eigenvalues of 1000 random structured perturbations  $A + D\Delta E$  are plotted, where  $D$  and  $E$  are constructed as in Section 2, and where  $\Delta$  is a diagonal matrix with random entries on its diagonal. We have scaled these entries, such that  $\|\Delta\| = 0.6$  and so the random perturbations yield a subset of the structured ( $\varepsilon = 0.6$ ) – pseudospectrum of  $A$ . As in part (a) we observe that the four eigenvalues close to the real axis are more sensitive to perturbations, in contrast to the situation in panel (b) of Figure 2.

It is not completely understood why our approach leads to this difference in the behaviour of the eigenvalues. However, the results demonstrate that it is important to choose the appropriate method for investigations. If one is interested in the sensitivity of eigenvalues of  $Q$  under perturbations of the full matrices  $M$ ,  $C$  and  $K$ , for instance because of uncertainties in the system, then matrix norms as used in (6) are the correct measure for the size of perturbations.

If, however, the behaviour of eigenvalues under variation of certain entries in the system’s matrix  $A$  should be investigated, for instance to understand the influence of parameters, which appear in certain positions in  $A$ , then an analysis that relies on matrix norms can give misleading results, as the above example shows. In this case the use of structured pseudospectra as defined above is more appropriate.

We note that similar differences in the sensitivity of eigenvalues have been observed in [18], where general and structured pseudospectra have been used in the analysis of an eccentric rotor fitted with an automatic dynamic balancer.

**Remark.** This paper deal with pseudospectra of single matrices with respect to perturbations of their elements. Therefore we have applied the approach of Section 2 to the matrix  $A$  of the first order system. The direct study of pseudospectra with respect to perturbations of elements of the matrices  $M$ ,  $C$  and  $K$  requires the analysis of structured pseudospectra of the matrix polynomial  $Q$ , generalising an approach presented in [5]. Note that similarly, perturbations of  $M$ ,  $C$  and  $K$  can be described by linear fractional perturbations of the matrix  $A$ , [19,?]. The corresponding study of structured pseudospectra for higher order systems shall be the subject of our future research.  $\square$

### 3.2 Vulnerability of truss structures

We finally return to the problem of analysing the vulnerability of a truss structure. As already described in the Section 1 we aim to derive a measure for

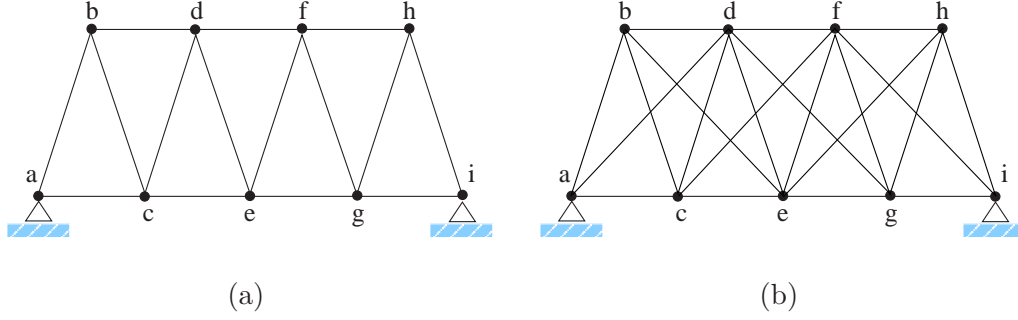


Fig. 4. Two simple planar truss-structures  $\mathcal{S}_{a,b}$ .

vulnerability that uses only information from the associated stiffness matrix  $K$ . Damage to a structure can be interpreted as a perturbation of the stiffness matrix and thus the distance to singularity of  $K$  can serve as a measure for vulnerability.

This mathematical approach will be tested on two examples of two-dimensional truss structures. Of course, similar considerations can be made for three-dimensional truss structures. We will compute the distance to singularity of the stiffness matrices using both their general and their structured pseudo-spectra.

Our first example concerns the two simple truss structures in Figure 4. Panel (a) contains a structure  $\mathcal{S}_a$ , which consists of 9 nodes that are connected by 15 members. The nodes **a** and **i** are fixed (as indicated by the triangles) and therefore not subject to displacements in horizontal or vertical direction, such that the whole structure has  $2 \times 7$  degrees of freedom. For the members connecting nodes **a**, **b** and **h**, **i** the material constants are assumed to be  $AE = 1100$  kN, whereas for all other members  $AE = 3000$  kN.

The relation between external forces and displacements of the nodes is given by Equation (2), and it is straightforward to compute the stiffness matrix  $K_a$ , which reads



$$K_a = 10^6 \times \begin{pmatrix} 1.9 & 0.3 & -0.3 & -0.5 & -1.5 & 0.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 1.5 & -0.5 & -1.1 & 0.0 & 0.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.3 & -0.5 & 3.5 & 0.0 & -0.3 & 0.5 & -1.5 & 0.0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & -1.1 & 0.0 & 2.1 & 0.5 & -1.1 & 0.0 & 0.0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.5 & 0.0 & -0.3 & 0.5 & 3.5 & 0.0 & -0.3 & -0.5 & -1.5 & 0.0 & 0 & 0 & 0 & 0 \\ 0.0 & 0.0 & 0.5 & -1.1 & 0.0 & 2.1 & -0.5 & -1.1 & 0.0 & 0.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.5 & 0.0 & -0.3 & -0.5 & 3.5 & 0.0 & -0.3 & 0.5 & -1.5 & 0.0 & 0 & 0 \\ 0 & 0 & 0.0 & 0.0 & -0.5 & -1.1 & 0.0 & 2.1 & 0.5 & -1.1 & 0.0 & 0.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.5 & 0.0 & -0.3 & 0.5 & 3.5 & 0.0 & -0.3 & -0.5 & -1.5 & 0.0 \\ 0 & 0 & 0 & 0 & 0.0 & 0.0 & 0.5 & -1.1 & 0.0 & 2.1 & -0.5 & -1.1 & 0.0 & 0.0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.5 & 0.0 & -0.3 & -0.5 & 3.5 & 0.0 & -0.3 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0 & 0.0 & -0.5 & -1.1 & 0.0 & 2.1 & 0.5 & -1.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.5 & 0.0 & -0.3 & 0.5 & 1.9 & -0.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0 & 0.0 & 0.5 & -1.1 & -0.3 & 1.5 \end{pmatrix};$$

see for instance [9] for how to assemble a stiffness matrix for a given structure. Note how the connectivity of  $\mathcal{S}_a$  is reflected in the sparsity structure of  $K_a$ .

Panel (b) of Figure 4 contains a similar structure  $\mathcal{S}_b$ . However, in this structure 6 additional members have been introduced. Furthermore, for all members in this structure  $AE = 1150$  kN. It is straightforward to derive the associated stiffness matrix  $K_b$ . We do not write it down.

Let us discuss the stability of the structures. For this one has to investigate how the structures have to be damaged in order for them to collapse. Structural analysis shows that  $\mathcal{S}_a$  will collapse if any of its diagonal members is completely damaged, i.e. removed. Because of the chosen material constants it should be easiest to remove the element connecting nodes **a**, **b** or the one connecting **h**, **i**. In contrast, the structure  $\mathcal{S}_b$  will not collapse if only one of its members is removed. Measured in terms of perturbations to material constants of members bigger damage is therefore necessary for a failure of  $\mathcal{S}_b$ . From this point of view, the structure should be more robust.

Now let us take a look at how this is reflected in the proposed measures for stability involving the stability matrices  $K_{a,b}$  and let us compute their distance to singularity. Since a matrix is singular if and only if 0 is contained in its spectrum, this amounts to computing the smallest  $\varepsilon$  such that 0 is contained in the  $\varepsilon$ -pseudospectrum of  $K_{a,b}$ .

We first consider general perturbations of  $K_{a,b}$ , that is we consider  $K_{a,b} + P$  with  $P \in \mathbb{C}^{14 \times 14}$  and we equip this space as usual with the 2-norm. Denoting the distance to singularity with respect to this norm by  $d_2$  we find that

$$d_2(K_{a,b}) := \min\{\|P\| : K_{a,b} + P \text{ is singular}\} = 1/\|K_{a,b}^{-1}\| = \lambda_{\min}(K_{a,b}), \quad (8)$$

where  $\lambda_{\min}$  denotes the smallest eigenvalue of a matrix, see [12,2]. Note that  $K_{a,b}$  are symmetric and positive-definite, so that all their eigenvalues are real and positive. For the truss structures in Figure 4 we find  $d_2(K_a) = 6.34 \times 10^4$  and  $d_2(K_b) = 6.15 \times 10^4$ . Hence, according to this measure  $\mathcal{S}_a$  is more stable, contradicting the considerations above.

Let us now consider structured perturbations of  $K$ . We will only deal with variations in the material of the members of the structure. Recalling the representation of the stiffness matrices  $K_{a,b} = V_{a,b}^T \cdot M_{a,b} \cdot V_{a,b}$ , where the diagonal matrix  $M$  contains all material information, we are thus led to considering perturbations of  $K_{a,b}$  of the following form:

$$K_{a,b} + V_{a,b}^T \cdot \Delta \cdot V_{a,b}, \quad \Delta = \text{diag}(d_1, \dots, d_m). \quad (9)$$

But these are just structured perturbations as discussed in Section 2 with  $D = V_{a,b}^T$ ,  $E = V_{a,b}$  and, in particular, Lemma 2.1 allows us to compute the pseudospectra of  $K_{a,b}$  with respect to this class of perturbations. We are interested in the distance to singularity, and in the notation of Lemma 2.1 we therefore have to determine

$$d_S(K_{a,b}) := F_{a,b}(0) = \mu_{\Delta} \left( -V_{a,b} \cdot K_{a,b}^{-1} \cdot V_{a,b}^T \right). \quad (10)$$

For the computation we use again MATLAB's routine `psv` routine and find that  $d_S(K_a) = 1.1 \times 10^6$ , as expected, because removing either of the elements with  $AE = 1100 \text{ kN}$  leads to a collapse of  $\mathcal{S}_a$ . Furthermore,  $d_S(K_b) = 1.15 \times 10^6$ , so that this measure confirms that structure  $\mathcal{S}_b$  is more robust against damage of its members than structure  $\mathcal{S}_a$ .

**Remark:** *It might be surprising that the difference between  $d_S(K_a)$  and  $d_S(K_b)$  is so marginal. This, however, is due to the fact that structured perturbations  $K_{a,b} + V_{a,b}^T \cdot \Delta \cdot V_{a,b}$  can yield matrices that are not stiffness matrices of structures. Indeed, the nearest singular matrix to  $K_b$  with respect to structured perturbations would correspond to a structure with members possessing negative material constants.*

This first example has been specifically designed to demonstrate the usefulness of structured perturbations. We will finally discuss an example, which shows that it is often useful to combine the information of the structured and unstructured approach. This time we deal with the three truss structures shown in Figure 5. In each of these structures we have for all vertical bars  $AE = 1950 \text{ kN}$ , while for all horizontal bars  $AE = 838 \text{ kN}$  and for the diagonal bars  $AE = 779 \text{ kN}$ .

We follow a similar procedure as in the first example and determine the dis-

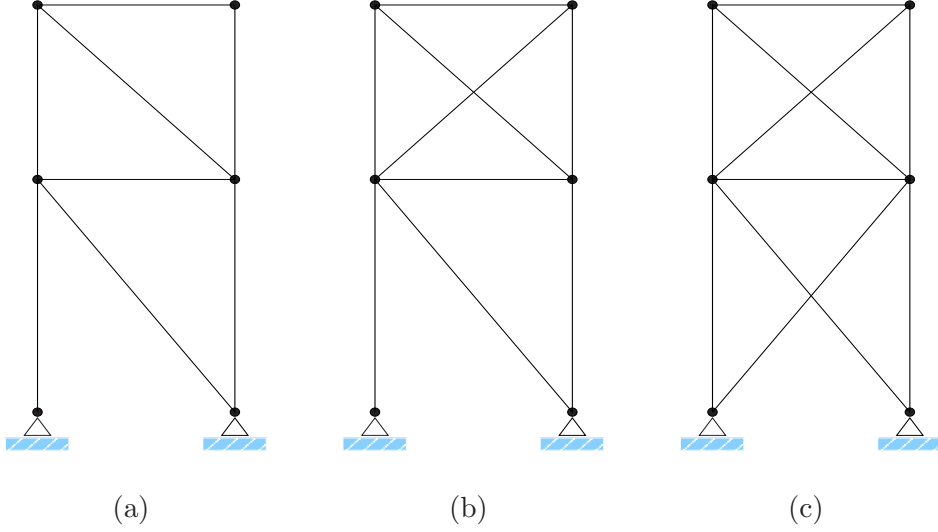


Fig. 5. Three planar truss-structures.

tance to singularity of the associated stiffness matrices  $K_a, K_b, K_c$ . (The index corresponds to the panel, in which the structure appears in Figure 5.) In order to determine the minimal amount of damage that has to be applied to the structure to achieve collapse perturbations of the form (9) are considered first. We find that for all three structures this measure of vulnerability is given by  $d_S = 779000$ . This is not surprising, since after removing the diagonal members, which have the smallest material constants, each of the structures becomes a mechanism.

It is evident that the structured approach does not distinguish the number of bars that have to be removed, because only the maximal size of the entry-wise perturbations is taken into account. A further graduation can be obtained by considering the distance to singularity of the stiffness matrices with respect to general perturbations. Determining the  $d_2$ -values for  $K_{a,b,c}$ , which amounts to computing the smallest eigenvalue of these matrices, we find that

$$d_2(K_a) = 7709, \quad d_2(K_b) = 8761, \quad d_2(K_c) = 14200.$$

Hence, this measure indicates that the structure in panel (a) is less robust than the one in panel (c). However, as the first example shows, one should not rely exclusively on the graduation of robustness according to the measure  $d_2$ .

The two examples show that both  $d_S$  and  $d_2$  contain valuable information about the underlying truss structure. The measure  $d_2$  contains global information about the stiffness matrix. It can be used for a rough description of vulnerability, but since it does not allow a direct physical interpretation, results can be misleading as the first example shows.

The measure  $d_S$  describes the minimal amount of damage that has to be ap-

plied to members in order to cause collapse. Thus, this measure is in agreement with the theory of vulnerability developed in [10], which identifies weak parts in a structure. We therefore propose to use  $d_S$  to describe the vulnerability of truss structures. (Recall, however, that the damaged structure could be non-physical with negative material constants, compare with the Remark on page 17.) If the vulnerability of several structures is compared and if their  $d_S$ -values agree as in our second example, then additional information can be obtained from the consideration of unstructured perturbations. As we have seen in this example, the distance  $d_2$  can then be related to the number of members that have to be removed.

## 4 Conclusions

In this article we have presented a new method for computing structured pseudospectra of a matrix with respect to perturbations that modify only certain of its entries. Particular attention has been paid to perturbations that preserve a given sparsity structure. The method relates the computation of pseudospectra to the computation of the ssv of an associated matrix. Since the computations mainly depend on the number of entries that are to be modified the method works well for medium-sized and highly sparse matrices.

The approach is also of interest for the sensitivity analysis of systems under variation of their parameters. Parameters usually enter calculations in certain positions of the system's equation or matrix, such that their variation only affects certain entries. This influence can be hidden if one works exclusively with matrix norms. By choosing a suitable weight matrix  $W$  it is more appropriate to compute structured pseudospectra. It has been demonstrated in the example of the wing problem that both methods yield indeed different results.

As a main application we considered the vulnerability analysis of truss structures, where we discussed measures of vulnerability of a structure, that can be derived using only information of the associated stiffness matrix. This approach allows one to derive a global measure for the structure in a straightforward way and can support a traditional or algorithmic vulnerability analysis. Using structured pseudospectra we derived a lower bound for the damage that has to be applied to members in a structure in order to cause collapse. This measure relates to the theory of vulnerability developed in [10], which detects weak parts of a structure.

The investigations about applying structured pseudospectra to the vulnerability analysis of truss structures are still at an early stage and future research will be devoted to improving the methods. For instance, the methods developed here are based on complex perturbations of the matrix. In applications, how-

ever, one usually deals with real perturbations only. It is interesting to see if the theory can be extended to the case of structured pseudospectra with respect to real perturbations. Relevant results for general pseudospectra have been derived in [20]. Furthermore, perturbations of the stiffness matrix of the form (9) are easier to interpret, if no negative values for material constants appear in the perturbed matrix, compare also with the Remark on page 17. At the moment it is not known if a computable formula for structured pseudospectra under this condition can be derived.

In this paper we have applied the concept of pseudospectra of a matrix to the static problem of vulnerability of truss structures. Formulae (8) and (10) show that in this case one particular piece of information has to be obtained from the pseudospectra of the stiffness matrix, namely its distance to singularity. In our future research we will consider the advanced dynamic problem of robustness of structures under vibrations. This problem requires the analysis of the behaviour of all the eigenvalues of the system, and therefore the full pseudospectrum has to be considered, similar to the discussion of the wing problem in Section 3.1. Moreover, the dynamic problem requires the analysis of second order differential equations, similar to Section 3.1. In this context, the analysis of structured pseudospectra for matrix polynomials becomes important and will be the subject of our future research.

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